SUPERSONIC PROFILES WITH MINIMUM DRAG

(O SVERKHZVUKOVYKH PROFILIAKH, IMEIUSHCHIKH MINIMAL'NOE SOPROTIVLENIYE)

PMM Vol.22, No.2, 1958, pp.269-273

Iu.D. SHMYGLEVSKII
(Moscow)

(Received 2 July 1958)

1. Integral formulae. Let us deal with the problem of finding a profil with minimum wave drag in a supersonic stream of gas. Assume a uniform gas flow with approach velocity w_∞ parallel to the x axis, and also point A and B through which the required profile is to pass (Fig.1).

Shock wave AC passes through point A, and in special cases it can merge (or degenerate) into the characteristic of the approaching gas stream.

Now draw through point B a characteristic of a second family which intersects the shock wave at C. From C draw the characteristic of the first family to its intersection on profile AB at point D.

The flow is determined by equations of continuity and of motion, by the Bernouilli integral and by an equation expressing the adiabatic law,

$$\frac{\partial \rho w \cos\vartheta}{\partial x} + \frac{\partial \rho w \sin\vartheta}{\partial y} = 0 \qquad (1.1)$$

$$w \cos\vartheta \, \frac{\partial w \cos\vartheta}{\partial x} + w \sin\vartheta \, \frac{\partial w \cos\vartheta}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad (1.2)$$

$$\frac{w^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \frac{1}{2}\frac{\varkappa + 1}{\varkappa - 1} \qquad (1.3)$$

$$\frac{p}{\rho^\varkappa} = f(\psi) \qquad (1.4)$$

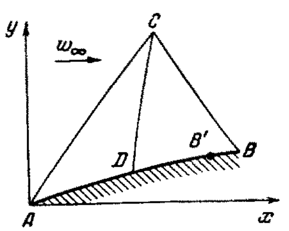

Fig. 1.

where x, y are cartesian coordinates, w is the velocity in terms of the critical flow velocity, a_*; θ is the angle of inclination of the velocity with respect to the x axis, ρ is the gas density in terms of the density of the approaching stream ρ_∞; p is the pressure in terms of $\rho_\infty a_*^{\,2}$, κ is the adiabatic index, ψ is a stream function.

We now introduce Mach angle α where $\sin^2\alpha = \kappa p/\rho w^2$.

If line CD is indeed a characteristic, the part of the profile BD does not influence flow to the left of CD. It therefore follows that the

Reprint Order No. PMM 33.

section BD should give minimum drag. Here the auxiliary problem arises of determining the shape of contour BD for the given characteristic CD. This problem is exactly analogous to that of axially symmetrical flow referred to in [1, 2], which is solved in the same way and leads to the following results.

On characteristic BC, the magnitudes of α and θ can be obtained from

$$\mu \sin 2\alpha + \varkappa [\mu \sin 2\vartheta + \lambda (1 - \cos 2\vartheta)] = 0 \tag{1.5}$$

$$\frac{S(\sigma) A(\alpha)}{\sqrt{\varkappa}} - \frac{\sin^2 \vartheta}{\mu \cos \alpha} \sqrt{\frac{\varkappa + 1}{\varkappa - \cos 2\alpha}} = 0 \tag{1.6}$$

where

$$A(\alpha) = \left(\frac{\varkappa + 1}{2\varkappa} \, \frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{-\frac{1}{2} \frac{\varkappa+1}{\varkappa-1}}$$

The quantity $\sigma(\psi)$ has the following significance. In what follows it will be inconvenient to introduce an entropy function. It is more convenient to assume that the gas has passed through shock wave AC, whose angle of inclination to the x axis for each value of ψ equals $\sigma(\psi)$. The quantity $S(\sigma)$ is given by

$$S(\sigma) = \left[\frac{2w_\infty^2 \sin^2 \sigma}{\varkappa + 1} - \frac{\varkappa - 1}{2\varkappa}(1 - W)\right]^K \left[\frac{1 - W \cos^2 \sigma}{w_\infty^2 \sin^2 \sigma}\right]^\varkappa$$

where

$$W = \frac{\varkappa - 1}{\varkappa + 1} w_\infty{}^2, \qquad K = \frac{1}{\varkappa - 1}$$

The quantities λ and μ which appear in (1.5) and (1.6) are constant Lagrange multipliers.

Now let us study the full problem.

Let us denote the contour formed by shock wave AC, the characteristic BC and profile AB by L and the region enclosed by this contour by Σ.

From equations (1.1), (1.2), it follows that

$$\frac{\partial}{\partial x}(p + \rho w^2 \cos^2 \vartheta) + \frac{\partial}{\partial y} \rho w^2 \sin \vartheta \cos \vartheta = 0$$

If we integrate both parts of this equation over region Σ, and transform by Green's formula to a contour integral, we obtain,

$$\oint_L - \rho w^2 \sin \vartheta \cos \vartheta \, dx + (p + \rho w^2 \cos^2 \vartheta) \, dy = 0 \tag{1.7}$$

The component χ of this integral along the line AB ($dy = \tan \theta \, dx$), is equal to the drag of the profile up to a multiplicative constant

$$\chi = \int_{y=y_A}^{y_B} p \, dy$$

Now express χ in terms of integrals along AC and BC, using (1.7). Along the shock wave we have:

$$dx = \frac{\operatorname{ctg}\sigma}{w_\infty}\,d\psi, \qquad dy = \frac{1}{w_\infty}\,d\psi \tag{1.8}$$

Moreover, to the left of the shock wave

$$\rho = 1, \quad w = w_\infty, \quad \vartheta = 0, \quad p = \frac{\varkappa - 1}{2\varkappa}\left(\frac{\varkappa + 1}{\varkappa - 1} - w_\infty{}^2\right)$$

Along the characteristic of the second family

$$dx = -S(\sigma)\,A(\alpha)\,\frac{\cos(\vartheta - \alpha)}{\sqrt{\varkappa}}\,d\psi, \quad dy = -S(\sigma)\,A(\alpha)\,\frac{\sin(\vartheta - \alpha)}{\sqrt{\varkappa}}\,d\psi \tag{1.9}$$

Making use of these equalities, the expression for a and equation (1.3) we obtain

$$\bar{\chi} = \int\limits_{\psi=0}^{\psi_c}\left\{\frac{\varkappa+1}{2\varkappa}\left(w_\infty + \frac{1}{w_\infty}\right) - \sqrt{\frac{\varkappa+1}{\varkappa - \cos 2\alpha}}\left[\cos\vartheta - \right.\right.$$

$$\left.\left. - \frac{1}{\varkappa}\sin\alpha\sin(\vartheta - \alpha)\right]\right\}d\psi \tag{1.10}$$

where $\chi = \rho_0\sqrt{(\kappa + 1)/2}\;\;\bar{\chi}$ (ρ_0 is the stagnation density).

The given quantities $X = x_B - x_A$ and $Y = y_B - y_A$ can, with the help of (1.8) and (1.9) also be expressed as integrals along AC and BC:

$$X = \int\limits_{\psi=0}^{\psi_c}\left[\frac{\operatorname{ctg}\sigma}{w_\infty} + \frac{1}{\sqrt{\varkappa}}\,S(\sigma)\,A(\alpha)\cos(\vartheta - \alpha)\right]d\psi$$

$$Y = \int\limits_{\psi=0}^{\psi_c}\left[\frac{1}{w_\infty} + \frac{1}{\sqrt{\varkappa}}\,S(\sigma)\,A(\alpha)\sin(\vartheta - \alpha)\right]d\psi \tag{1.11}$$

2. Variational Problem. The following variational problem arises. For given w_∞, X and Y, find the function $\sigma(\psi)$ which gives a minimum of the function (1.10) for iso-primetric conditions (1.11), if $a(\sigma)$ and $\theta(\sigma)$ are determined by the system of equations (1.5) and (1.6). The latter condition may not be fulfilled when a boundary extremum condition prevails.

From the fact that shock waves can exist in ABC follows the admissibility of "piecewise" continuous functions $a(\sigma)$ and $\theta(\sigma)$ which satisfy the shock relations at the points of discontinuity. The permissible functions $\sigma(\psi)$ are continuous functions.

We will use the Lagrange multiplier method to solve the problem. Let us set up the functional

$$I = \int\limits_{\psi=0}^{\psi_c}\Phi(\sigma)\,d\psi \tag{2.1}$$

where

$$\Phi(\sigma) = \frac{\varkappa+1}{2\varkappa}\left(w_\infty + \frac{1}{w_\infty}\right) - \sqrt{\frac{\varkappa+1}{\varkappa - \cos 2\alpha}}\left[\cos\vartheta - \frac{1}{\varkappa}\sin\alpha\sin(\vartheta-\alpha)\right] +$$

$$+ \frac{\lambda + \mu\,\mathrm{ctg}\,\sigma}{w_\infty} + \frac{1}{\sqrt{\varkappa}} S(\sigma)\, A(\alpha)\,[\lambda\sin(\vartheta-\alpha) + \mu\cos(\vartheta-\alpha)]$$

and λ, μ are Lagrange multipliers which have to be determined. Their values can be determined from condition (1.11). Obviously equations (1.5) and (1.6) are still valid and determine the dependence of $a(\sigma)$ and $\theta(\sigma)$.

The first variation of function (2.1) takes the following form;

$$\delta I = \Phi\,|_{\psi=\psi_c}\delta\psi_c + \int\limits_{\psi=0}^{\psi_c} \Phi_\sigma \delta\sigma d\psi$$

In order to obtain the extremum we equate the expression in front of the integral sign to zero.

$$\Phi\,|_{\psi=\psi_c} = 0 \tag{2.2}$$

Inasmuch as permissible variations δX and δY are zero, we obtain

$$\delta\overline{\chi} = \int\limits_{\psi=0}^{\psi_c} \Phi_\sigma \delta\sigma d\psi \tag{2.3}$$

In order to obtain the extremum the integrand in (2.3) should be equated to zero. This gives $\Phi_\sigma = 0$, or

$$w_\infty \sin^2\sigma\cos\sigma S(\sigma)\, s(\sigma)\, A(\alpha)\,[\lambda\sin(\vartheta-\alpha) + \mu\cos(\vartheta-\alpha)] - \sqrt{\varkappa}\,\mu = 0. \tag{2.4}$$

where

$$s(\sigma) = \frac{2\varkappa}{\varkappa-1}\left[\frac{4w_\infty{}^2\sin\sigma}{4\varkappa w_\infty{}^2\sin^2\sigma - (\varkappa^2-1)(1-W)} - \frac{1-W}{(1-W\cos^2\sigma)\sin\sigma}\right]$$

Apart from a, θ and σ equations (1.5), (1.6) and (2.4) only contain constant magnitudes. It follows therefore that in fulfilling the Euler equation (2.4) we obtain a = const. θ = const. σ = const.

It might appear that the solution of the problem is simple in form; the shock wave is a straight line, behind the wave there is uniform flow and the required profile is a straight line. However this is not so.

Eliminating λ and μ from (1.5) and (2.4), we then get

$$\varepsilon = s(\sigma)\, A(\alpha)\,(\sin 2\alpha + \varkappa\sin 2\vartheta) +$$

$$+ \frac{2\varkappa\sin^2\vartheta}{\sin(\vartheta-\alpha)}\left[\frac{\sqrt{\varkappa}}{w_\infty S(\sigma)\sin^2\sigma\cos\sigma} - s(\sigma)\, A(\alpha)\cos(\vartheta-\alpha)\right] = 0 \tag{2.5}$$

If we now put the values of $a(\sigma, w_\infty)$, $\theta(\sigma, w_\infty)$, known from the shock wave expressions, into (2.5), it is easy to see that the equation is not satisfied identically.

One of the roots of equation $\epsilon = 0$ is $\sigma = \text{arc sin } M_\infty^{-1}$, where M_∞ is the Mach number of the approaching flow.

For any thickness ratio $l = X : Y$ a solution which satisfies (2.4) and the shock wave expressions, is in general not possible. The solution must be found from particular physically attainable flows. The problem is indeed that of connecting the straight line portion of the shock wave with the straight line characteristic by lines of the shock wave and of the characteristic of the second family which give the boundary extremum. In the case of rarefaction flows such lines are those determined by a break in the contour *AB*.

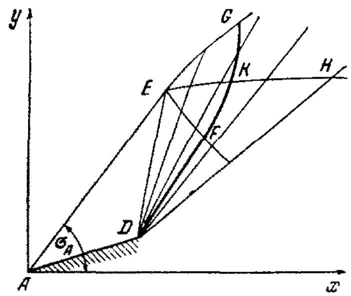

Fig. 2.

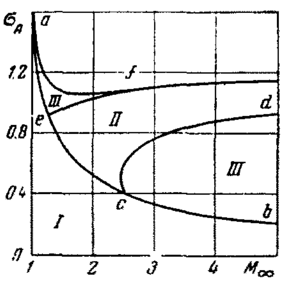

Fig. 3.

The following cases are possible.

(a) The value $\epsilon = 0$ is attained at some point on characteristic *BC* in the case of an expansion flow (convex profile).

(b) Value $\epsilon = 0$ is attained in a compression flow (concave profile).

Let us discuss the first case. Let us construct all the possible flows which will satisfy the above conditions and correspond to various thickness ratios l. Here one linear dimension may be fixed. Suppose point *D* (Fig.2) represents the point of discontinuity on the profile causing expansion of the flow, *AE* is the straight line portion of the shock wave, *DE* is the straight line characteristic. From point *D* there emanates a fan of characteristics of the first family. *EF* is the characteristic of the second family going through point *E*. *EKH* is the streamline passing through *E*. Such a flow is determined by one parameter, for instance σ_A.

Within the field of flow let us take note of line *DFKG*, which defines the geometric locus of points which embody the following properties. Equation $\epsilon = 0$ is fulfilled along *DFK*. Points in section *KG* are such that, if second-family-characteristics satisfying equations (1.5) and (1.6) are radiated from them, then at the points where these characteristics intersect the streamline *EH*, and therefore for smaller values of ψ, this equation $\epsilon = 0$ is again fulfilled. These characteristics and those of the first family from *FKG*, determine the required profile (Goursat's problem).

For such a flow it is not possible to find an explicit expression of the transverse condition analogous to (2.2). We will therefore use a numerical method for finding the minimum from the boundary conditions, i.e. for each σ_A we will evolve the attained profile with a given thickness ratio and then find the minimum of χ : Y with respect to σ_A numerically.

Note: In this problem the boundary extremum is realised and the first variation $\delta\chi$ in the solution is not zero. The generally accepted presumption that $\delta\chi \neq 0$ cannot give a solution to the problem appears to be incorrect.

TABLE 1

l	8.8590	5.0514	3.3197	2.5066
σ_A	0.42522	0.50039	0.60074	0.70141
x_B	0.99369	0.98096	0.95750	0.92881
y_B	0,11217	0.19420	0.28844	0.37055
x_D	0.10892	0.17266	0.23290	0.28030
y_D	0.01245	0.03478	0.07143	0.11323
$c_{x\ \min}$	0.09697	0,19401	0.33882	0.49994
$c_{x\ \text{break}}$	0.09698	0.19403	0.33888	0.49999

3. Examples. Fig. 3 shows the relation $\sigma_A(M_\infty)$ which satisfies $\epsilon = 0$. Line *ab* is determined by the equation;

$$\sigma = \arcsin \frac{1}{M_\infty}$$

whilst line *cd* gives the second root of $\epsilon = 0$, and line *ef* the third. Line *af* corresponds to a sonic velocity beyond the shock wave. Region I is not physically possible (a rarefaction shock), region II corresponds to concave profiles while region III is convex. Results of calculations made for $M_\infty = 3$ are shown in Table 1, and the profiles are shown in Fig. 4.

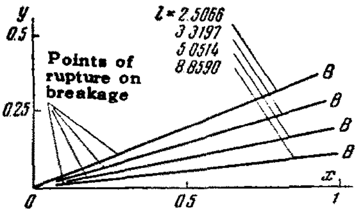

Fig. 4.

c_x is the coefficient of wave resistance. From Table 1 and Fig. 4 it is clear that minimum drag profiles, that is for the calculated cases, are to all intents and purposes wedge shaped, whilst any gain in c_x is negligible. The group of characteristics occupies only a small angle; the profile consists of two straight sections AD and DB'. The curved portion of profile $B'B$ is very small indeed. For example for profiles of thickness ratio $l = 2.5966$, after finding $c_{x\min}$, the values of $c_x(\sigma_A)$ shown in Table 2 were obtained.

The value $\sigma_A = 0.7000$ corresponds to the case when characteristic BC goes through point F (Fig. 2).

TABLE 2

σ_A	c_x	σ_A	c_x
		0.7010	0.499936
0.6980	0.499993	0.7012	0.499935
0.6990	0.499979	0.7014	0.499935
0.7000	0.499954	0.7016	0.499935
0.7002	0.499949	0.7018	0.499936
0.7004	0.499945	0.7020	0.499938
0.7006	0.499941	0.7040	0.499980
0.7008	0.499938	0.7080	0.500109

BIBLIOGRAPHY

1. Shmyglevskii, Iu.D., Nekotoryye variatsionnyye zadachi gazovoi dinamik osesimmetrichnykh sverkhzvukovykh techenii (Some variational problems in the gas dynamics of axially symmetrical supersonic flows). *PMM* Vol. 21, No. 2, 1957.

2. Shmyglevskii, Iu.D., Variatsionnaia zadacha gazodinamiki osesimmetrichnykh sverkhzvukovykh techenii (A variational problem in the gas dynamics of axially symmetrical supersonic flow). *Dokl. Akad. Nauk SSSR* Vol. 113, No. 3, 1957.

Translated by V.H.B.